Ex Helix $\underline{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$

$$\frac{d\underline{r}}{dt} = \underline{v} = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\,\hat{k}$$

$$\frac{d\underline{v}}{dt} = \underline{a} = -a\cos t\,\hat{i} + -a\sin t\,\hat{j} + 0\,\hat{k}$$

So $|\underline{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2+b^2}$ = speed

$$|\underline{a}| = \sqrt{(-a\cos t)^2 + (-a\sin t)^2} = \sqrt{a^2} = |a|$$

$$\hat{T} = \frac{\underline{v}}{|\underline{v}|} = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k}\right)$$

Tangent

- To calculate κ, will need $\left|d\hat{T}/dt\right|$. So...

$$\frac{d\hat{T}}{dt} = \frac{1}{\sqrt{a^2+b^2}}\left(-a\cos t\ \hat{i} - a\sin t\ \hat{j} + 0\hat{k}\right)$$

So

$$\left|\frac{d\hat{T}}{dt}\right| = \frac{a}{\sqrt{a^2+b^2}}$$

Then $\kappa = \left|\frac{d\hat{T}}{ds}\right| = \left|\frac{d\hat{T}}{dt}\frac{dt}{ds}\right| = \left|\frac{d\hat{T}}{dt}\right|\frac{1}{|\underline{v}|}$

$$\kappa = \frac{a}{\sqrt{a^2+b^2}} \cdot \frac{1}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2} \qquad \text{Curvature}$$

- $$\hat{N} = \frac{\frac{d\hat{T}}{ds}}{\left|\frac{d\hat{T}}{ds}\right|} = \frac{d\hat{T}/dt}{\left|d\hat{T}/dt\right|} = \frac{\frac{1}{\sqrt{a^2+b^2}}\left(-a\cos t\ \hat{i} - a\sin t\ \hat{j} + 0\hat{k}\right)}{\frac{a}{\sqrt{a^2+b^2}}}$$

$$\hat{N} = -\cos t\ \hat{i} - \sin t\ \hat{j} + 0\hat{k} \qquad \text{Normal}$$

- $$\hat{B} = \hat{T} \times \hat{N} = \frac{1}{\sqrt{a^2+b^2}}\left(b\sin t\ \hat{i} - b\cos t\ \hat{j} + a\hat{k}\right) \qquad \text{Bi normal}$$
- To calculate torsion:

$$\tau = -\frac{d\hat{B}}{ds} \circ \hat{N} = -\frac{d\hat{B}}{dt}\frac{dt}{ds} \circ \hat{N} = \frac{-1}{|\underline{v}|}\frac{d\hat{B}}{dt} \circ \hat{N}$$

$$\frac{d\bar{B}}{dt} = \frac{1}{\sqrt{a^2+b^2}}\left(b\cos t\ \hat{i} + b\sin t\ \hat{j} + 0\hat{k}\right)$$

$$\tau = \frac{-1}{|\underset{\sim}{v}|} \frac{d\hat{B}}{dt} \cdot \hat{N}$$

$$= \frac{-1}{\sqrt{a^2+b^2}} \left(\frac{1}{\sqrt{a^2+b^2}}\right)\left(b\cos t\, \hat{i} + b\sin t\, \hat{j} + 0\hat{k}\right) \cdot \left(-\cos t\, \hat{i} - \sin t\, \hat{j} + 0\hat{k}\right)$$

$$= \frac{-1}{a^2+b^2}\left(-b\cos^2 t - b\sin^2 t\right) = \frac{b}{a^2+b^2} \longleftarrow \text{torsion}$$

- To calculate $\underset{\sim}{a} = -a\cos t\, \hat{i} + -a\sin t\, \hat{j} + 0\hat{k}$

$$= \frac{d^2 s}{dt^2}\hat{T} + \kappa\left(\frac{ds}{dt}\right)^2 \hat{N}$$

$$= \frac{d|\underset{\sim}{v}|}{dt}\hat{T} + \kappa|\underset{\sim}{v}|^2 \hat{N}$$

Recall $|\underset{\sim}{v}| = \sqrt{a^2+b^2}$

$$\therefore \quad \underset{\sim}{a} = 0\,\hat{T} + \left(\frac{a}{a^2+b^2}\right)\left(\sqrt{a^2+b^2}\right)^2 \hat{N}$$

$$\underline{a} = a\hat{N} + 0\,\hat{T}$$

$$\hat{T} = \frac{d\underline{r}}{ds} = \frac{\underline{v}}{|\underline{v}|}$$

$$\underline{v} = \frac{d\underline{r}}{dt} = |\underline{v}| \cdot \hat{T}$$

$$\hat{N} = \frac{d\hat{T}/ds}{|d\hat{T}/ds|} = \frac{d\hat{T}/dt}{|d\hat{T}/dt|} = \frac{d\hat{T}/ds}{\kappa}$$

where $\kappa = \left|\frac{d\hat{T}}{ds}\right| = \left|\frac{d\hat{T}}{dt}\right| \frac{1}{|\underline{v}|} = \frac{a_N}{|\underline{v}|^2}$

$$\hat{B} = \hat{T} \times \hat{N}$$

$$\frac{d\hat{B}}{ds} = (-\tau)\hat{N} \Rightarrow \tau = \frac{-d\hat{B}}{ds} \circ N = \frac{-1}{|\underline{v}|}\frac{d\hat{B}}{dt} \circ \hat{N}$$

$$\underline{a} = \kappa\left(\frac{ds}{dt}\right)^2 \hat{N} + \frac{d^2 s}{dt^2} \hat{T}$$

$$= \kappa |\underline{v}|^2 \hat{N} + \frac{d|\underline{v}|}{dt} \hat{T} = a_N \hat{N} + a_T \hat{T}$$

$$a_N = \sqrt{|\underline{a}|^2 - a_T^2} = \kappa|\underline{v}|^2 \Rightarrow \kappa = \frac{a_N}{|\underline{v}|^2}$$